Math 217.003 F25
Quiz 12 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   > **Solution:** A function $T : V \to W$ such that for all $u, v \in V$ and all scalars α, β (in the underlying field $\mathbb{F}$ (e.g. $\mathbb{F} = \mathbb{R}$ the set of real numbers)),
   >
   > $$T(\alpha u + \beta v) = \alpha\, T(u) + \beta\, T(v).$$
   >
   > Equivalently, $T(u + v) = T(u) + T(v)$ and $T(\alpha v) = \alpha T(v)$ for all $u, v \in V$, $\alpha \in \mathbb{F}$.

   (b) To say that a list of vectors $(x_1, x_2, \ldots, x_d)$ in a vector space X is *linearly independent* means ...

   > **Solution:** That the only scalars $a_1, \ldots, a_d \in \mathbb{F}$ satisfying
   >
   > $$a_1 x_1 + \cdots + a_d x_d = 0_X$$
   >
   > are $a_1 = \cdots = a_d = 0$. Equivalently, no x_j can be written as a linear combination of the others.

   (c) Suppose U is a vector space and $u_1, \ldots, u_n \in U$. The list $(u_1, \ldots, u_n)$ is a *basis* for U provided that ...

   > **Solution:** (i) it is linearly independent, and (ii) it spans U, i.e.
   >
   > $$U = \operatorname{span}\{u_1, \ldots, u_n\} = \Big\{ \sum_{i=1}^{n} \alpha_i u_i : \ \alpha_i \in \mathbb{F} \Big\}.$$
   >
   > Equivalently: every $u \in U$ can be written *uniquely* as $u = \sum_{i=1}^{n} \alpha_i u_i$.

2. Let A be a $d \times n$ matrix with columns $\vec{C}_1, \ldots, \vec{C}_n$. Prove that the following are equivalent:

   (i) $[a_1 \ \ a_2 \ \ \ldots \ \ a_n]^\top$ is a solution to $A\vec{x} = 0$.

   (ii) $[a_1 \ \ a_2 \ \ \ldots \ \ a_n]^\top$ is in the kernel of the transformation $T_A : \mathbb{F}^n \to \mathbb{F}^d$ given by $T_A(\vec{x}) = A\vec{x}$.

   (iii) $a_1\vec{C}_1 + a_2\vec{C}_2 + \cdots + a_n\vec{C}_n = 0$ is a (nontrivial) linear relation among the columns.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: (i)$\Leftrightarrow$(ii): By definition, $\ker(T_A) = \{\vec{x} \in \mathbb{F}^n : T_A(\vec{x}) = A\vec{x} = 0\}$. Thus $[a_1 \ \dots \ a_n]^\top \in \ker(T_A)$ if and only if it solves $A\vec{x} = 0$.

(i)$\Leftrightarrow$(iii): Write $A = [\vec{C}_1 \ \vec{C}_2 \ \dots \ \vec{C}_n]$. For any $\vec{x} = [x_1 \ \dots \ x_n]^\top$,

$$A\vec{x} = x_1\vec{C}_1 + x_2\vec{C}_2 + \dots + x_n\vec{C}_n.$$

Therefore $A[a_1 \ \dots \ a_n]^\top = 0$ if and only if $a_1\vec{C}_1 + \dots + a_n\vec{C}_n = 0$. This is precisely the asserted column relation.

Since (i)$\Leftrightarrow$(ii) and (i)$\Leftrightarrow$(iii), all three statements are equivalent.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) Let A be a 71×8 matrix with columns $\vec{C}_1, \dots, \vec{C}_8$. Suppose the system $A\vec{x} = 0$ has solution $[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8]^\top$. The columns of A are linearly dependent.

   Solution: TRUE. Since $\vec{x} \neq 0$ solves $A\vec{x} = 0$, we have

   $$1\,\vec{C}_1 + 2\,\vec{C}_2 + 3\,\vec{C}_3 + 4\,\vec{C}_4 + 5\,\vec{C}_5 + 6\,\vec{C}_6 + 7\,\vec{C}_7 + 8\,\vec{C}_8 = 0,$$

   a nontrivial linear relation among the columns. Hence the columns are linearly dependent.

   (b) The list $\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -2 \end{bmatrix} \right)$ of vectors in $\mathbb{R}^3$ is linearly independent.

   Solution: FALSE. In fact,

   $$\begin{bmatrix} 6 \\ -2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix},$$

   so

   $$-2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   is a nontrivial linear relation. Therefore the three vectors are linearly dependent.